DEFLECTION OF FRAMES BY VIRTUAL WORK METHOD

Application of the virtual work method to determine the slopes and deflections of frames is similar to that for beams. To determine the deflection, Δ, or rotation, θ, at a point of a frame, a virtual unit load or unit couple is applied at that point. When the virtual system is subjected to the deformations of the frame due to real loads, the virtual external work performed by the unit load or the unit couple is $W_{ve} = 1(\Delta)$, or $W_{ve} = 1(\theta)$. As portions of the frame may undergo axial deformations in addition to the bending deformations, the total virtual internal work done on the frame is equal to the sum of the internal virtual work due to bending and that due to axial deformations. As discussed in the preceding section, when the real and virtual loadings and the flexural rigidity EI are continuous over a segment of the frame, the virtual internal work due to bending for that segment can be obtained by integrating the quantity $M_v M/EI$ over the length of the segment. The virtual internal work due to bending for the entire frame can then be obtained by summing the work for the individual segments; that is,

$$W_{vib} = \sum \int \frac{M_v M}{EI} dx \tag{7.32}$$

Similarly, if the axial forces F and F_v due to the real and virtual loads, respectively, and the axial rigidity AE are constant over the length L of a segment of the frame, then, as discussed in Section 7.3, the virtual internal work for that segment due to axial deformation is equal to $F_v(FL/AE)$. Thus, the virtual internal work due to axial deformations for the entire frame can be expressed as

$$W_{via} = \sum F_v \left(\frac{FL}{AE}\right) \tag{7.33}$$

By adding Eqs. (7.32) and (7.33), we obtain the total internal virtual work for the frame due to both bending and axial deformations as

$$W_{vi} = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx \tag{7.34}$$

By equating the virtual external work to the virtual internal work, we obtain the expressions for the method of virtual work for deflections and rotations of frames, respectively, as

$$1(\Delta) = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx \tag{7.35}$$

and

$$1(\theta) = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx \tag{7.36}$$

The axial deformations in the members of frames composed of common engineering materials are generally much smaller than the bending deformations and are, therefore, usually neglected in the analysis. In this text, unless stated otherwise, we will neglect the effect of axial deformations in the analysis of frames. The virtual work expressions considering only the bending deformations of frames can be obtained by simply omitting the first term on the right-hand sides of Eqs. (7.35) and (7.36), which are thus reduced to

$$1(\Delta) = \sum \int \frac{M_v M}{EI} dx \tag{7.37}$$

and

$$1(\theta) = \sum \int \frac{M_v M}{EI} dx \tag{7.38}$$

PROCEDURE FOR ANALYSIS

The following step-by-step procedure can be used to determine the slopes and deflections of frames by the virtual work method.

1. ***Real System*** Determine the internal forces at the ends of the members of the frame due to the real loading by using the procedure described in Section 5.6.
2. ***Virtual System*** If the deflection of the frame is to be determined, then apply a unit load at the point and in the direction of the desired deflection. If the rotation is to be calculated, then apply a unit couple at the point on the frame where the rotation is desired. Determine the member end forces due to the virtual loading.
3. If necessary, divide the members of the frame into segments so that the real and virtual loads and EI are continuous in each segment.
4. For each segment of the frame, determine an equation expressing the variation of the bending moment due to real loading (M) along the length of the segment in terms of a position coordinate x.
5. For each segment of the frame, determine the equation for the bending moment due to virtual load or couple (M_v) using the same x coordinate that was used for this segment in step 4 to establish the expression for the real bending moment, M. Any convenient sign convention can be used for M and M_v. However, it is important that the sign convention be the same for both M and M_v in a particular segment.
6. If the effect of axial deformations is to be included in the analysis, then go to step 7. Otherwise, determine the desired deflection or rotation of the frame by applying the appropriate virtual work expression, Eq. (7.37) or Eq. (7.38). End the analysis at this stage.
7. If necessary, divide the members of the frame into segments so that the real and virtual axial forces and AE are constant in each segment. It is not necessary that these segments be the same as those used in step 3 for evaluating the virtual internal work due to bending. It is important, however, that the same sign convention be used for both the real axial force, F, and the virtual axial force, F_v, in a particular segment.
8. Determine the desired deflection or rotation of the frame by applying the appropriate virtual work expression, Eq. (7.35) or Eq. (7.36).

EXAMPLES

1)Determine the rotation of joint C of the frame shown by the virtual work method

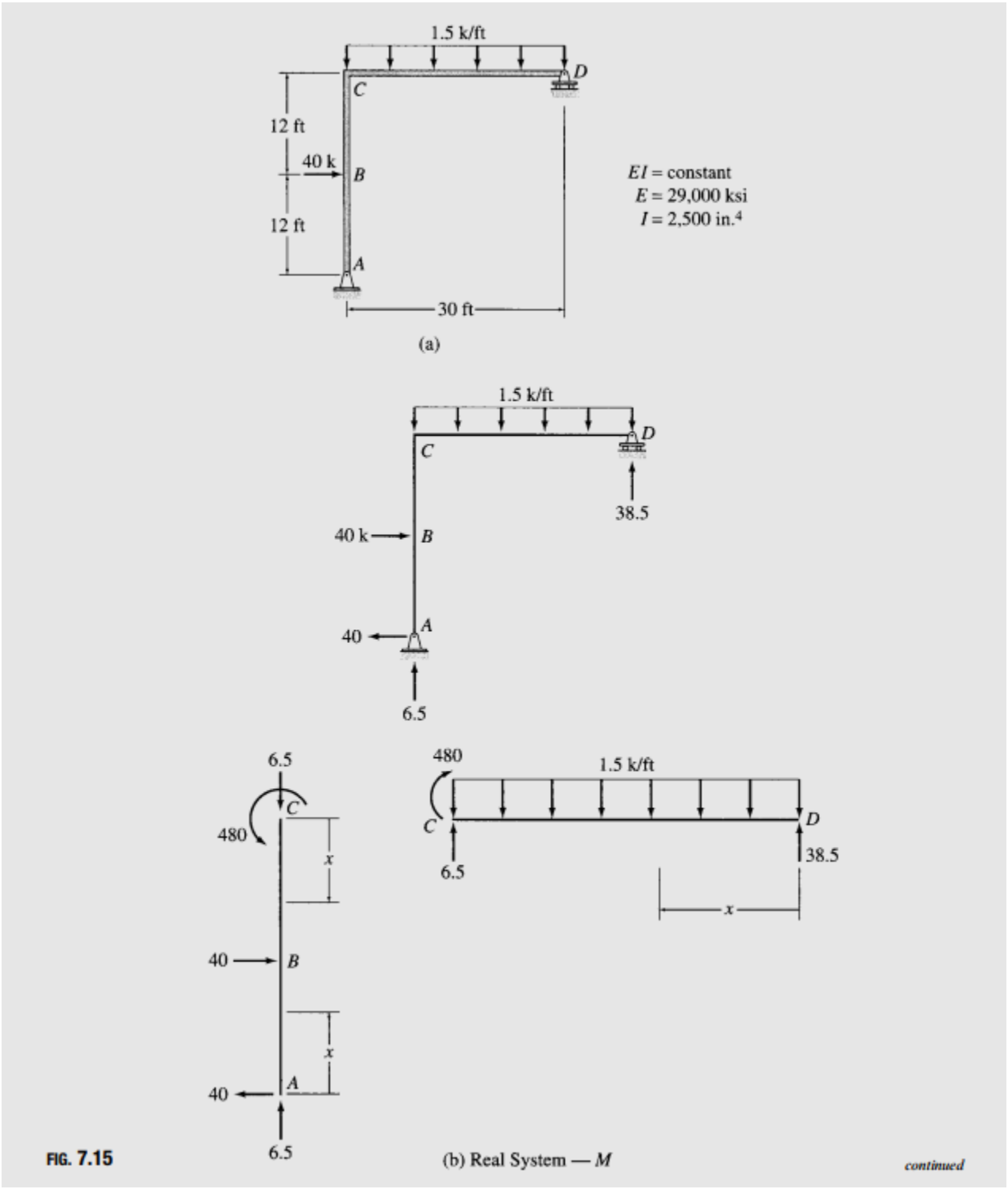
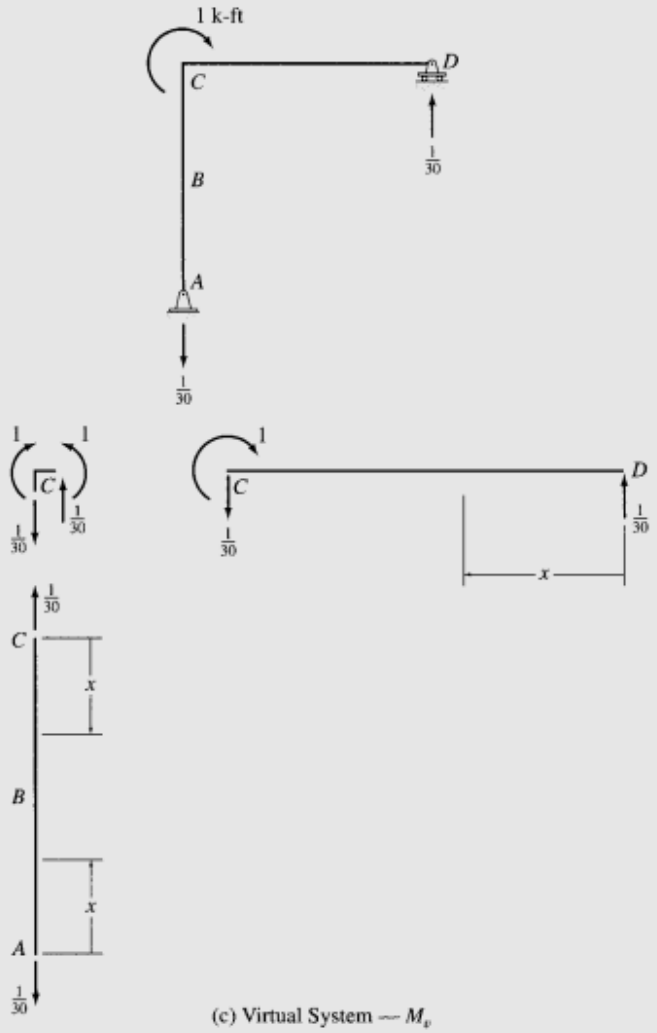


FIG. 7.15

continued



(c) Virtual System — M_v

FIG. 7.15 (contd.)

TABLE 7.8

	x Coordinate			
Segment	Origin	Limits (ft)	M (k-ft)	M_v (k-ft)
AB	A	0–12	$40x$	0
CB	C	0–12	480	0
DC	D	0–30	$38.5x - 1.5\frac{x^2}{2}$	$\frac{x}{30}$

$$1(\theta_C) = \sum \int \frac{M_v M}{EI} dx$$

$$= \frac{1}{EI} \int_0^{30} \left(\frac{x}{30}\right)\left(38.5x - 1.5\frac{x^2}{2}\right) dx$$

$$(1 \text{ k-ft})\theta_C = \frac{6{,}487.5 \text{ k}^2\text{-ft}^3}{EI}$$

Therefore,

$$\theta_C = \frac{6{,}487.5 \text{ k-ft}^2}{EI} = \frac{6{,}487.5(12)^2}{(29{,}000)(2{,}500)} = 0.0129 \text{ rad.}$$

$\theta_C = 0.0129$ rad. ↻ Ans.

2) Determine the vertical deflection of joint C

Solution

The real and virtual systems are shown in Figs. 7.16(b) and (c), respectively. The x coordinates used for determining the bending moment equations for the two members of the frame, AB and BC, are also shown in the figures. The equations for M and M_v obtained for the two members are tabulated in Table 7.9. The vertical deflection at joint C of the frame can now be calculated by applying the virtual work expression given by Eq. (7.37):

$$1(\Delta_C) = \sum \int \frac{M_v M}{EI} dx$$

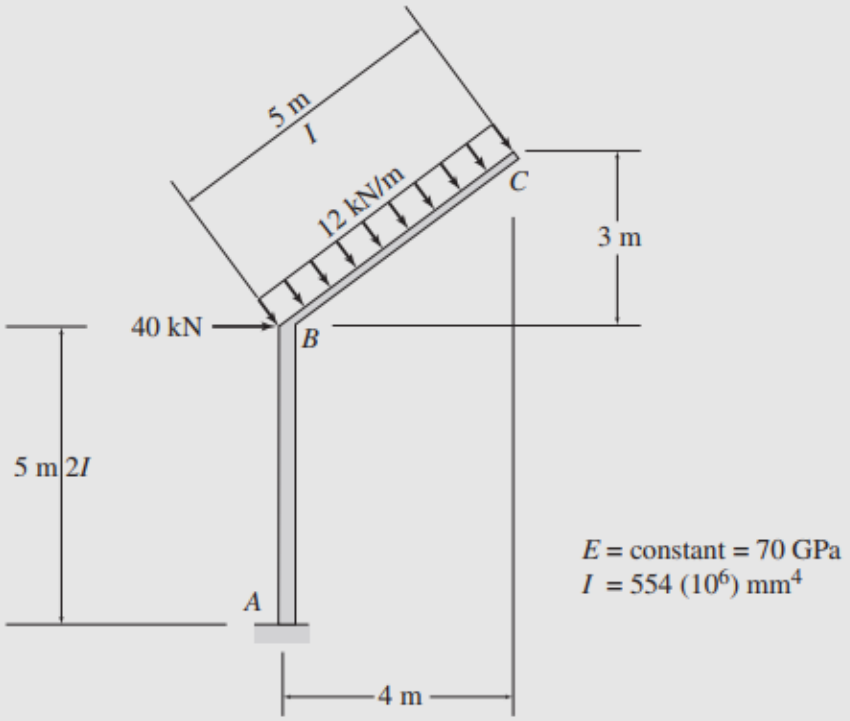
$$1(\Delta_C) = \frac{1}{EI}\left[\frac{1}{2}\int_0^5 (-4)(76x - 530)\,dx + \int_0^5 \left(-\frac{4}{5}x\right)(-6x^2)\,dx\right]$$

$$(1\text{ kN})\Delta_C = \frac{4{,}150\text{ kN}^2 \cdot \text{m}^3}{EI}$$

Therefore,

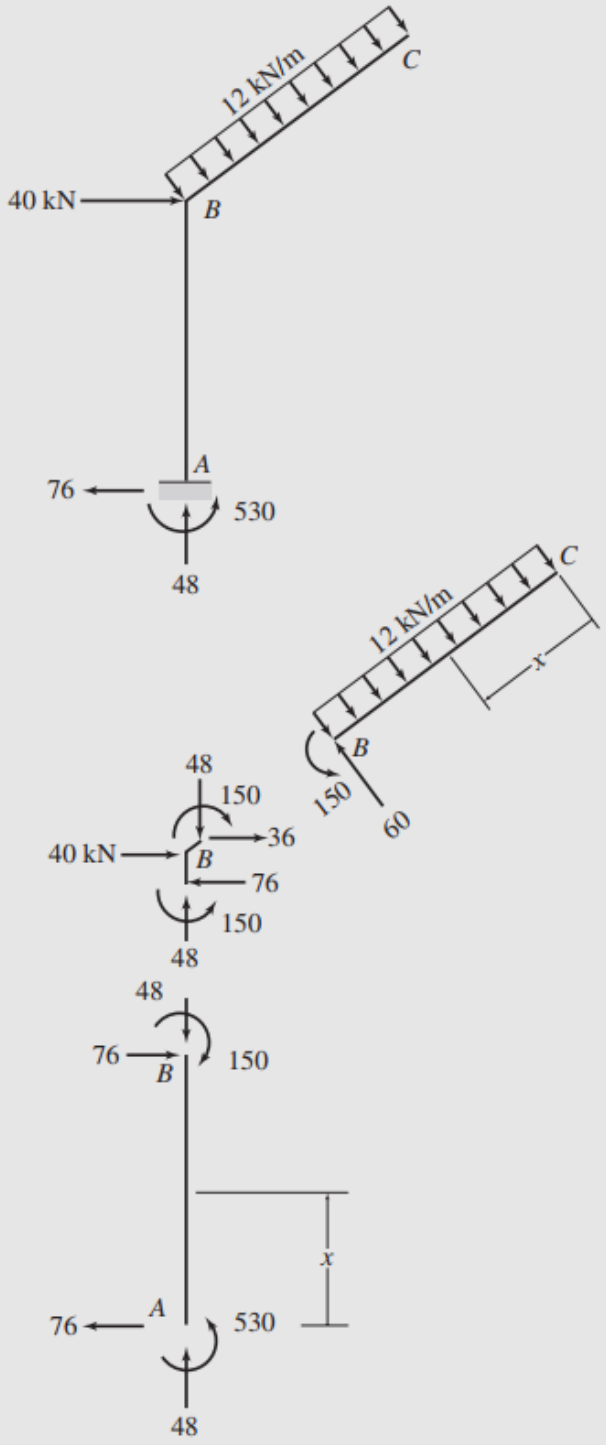
$$\Delta_C = \frac{4{,}150\text{ kN} \cdot \text{m}^3}{EI} = \frac{4{,}150}{70(554)} = 0.107\text{ m} = 107\text{ mm}$$

$$\Delta_C = 107\text{ mm} \downarrow$$ Ans.

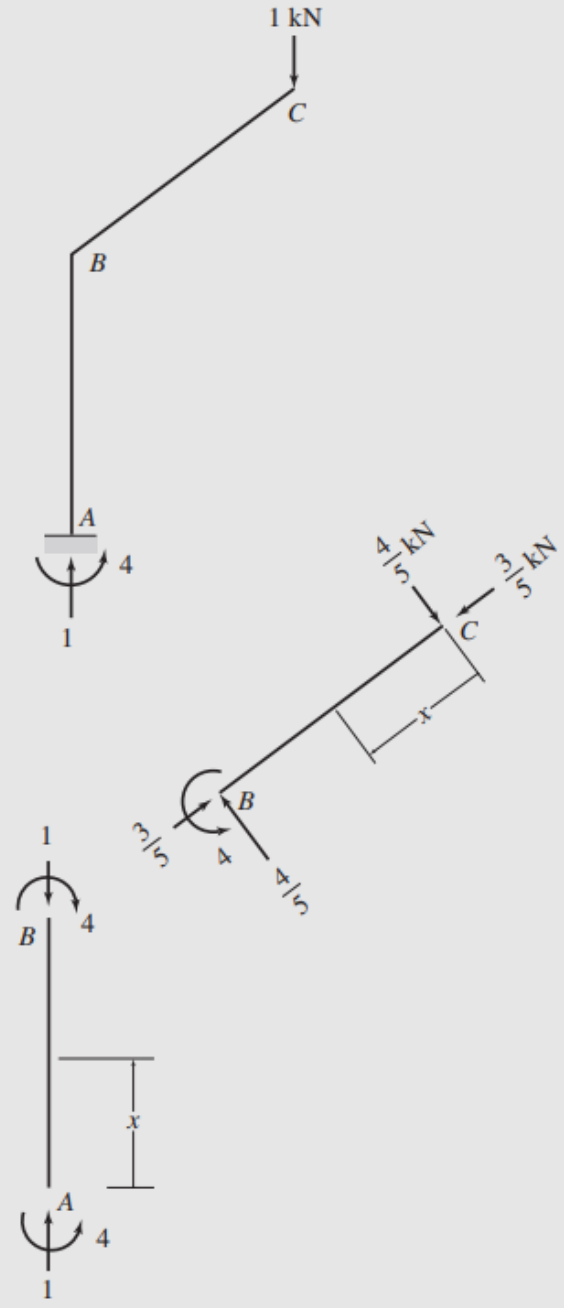


(a)

FIG. 7.16



(b) Real System — M



(c) Virtual System — M_v

TABLE 7.9

Segment	x Coordinate: Origin	x Coordinate: Limits (m)	EI ($I = 554 \times 10^6$ mm^4)	M (kN · m)	M_v (kN · m)
AB	A	0–5	$2EI$	$76x - 530$	-4
CB	C	0–5	EI	$-12\frac{x^2}{2}$	$-\frac{4}{5}x$